1.3.1 Governing Equations for Incompressible Flows in Primitive Form

**Primitive variables** are $\vec{V}$, $p$ and $T$. For incompressible flow of an isotropic, Newtonian fluid governing equations are

$$ \nabla \cdot \vec{V} = 0 $$  \hfill (1.7)

$$ \rho \frac{D\vec{V}}{Dt} = \rho \left( \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} \right) = -\nabla p + \nabla \cdot \left[ \mu \left( \nabla \vec{V} + \nabla \vec{V}^T \right) \right] + \rho \vec{f} $$  \hfill (1.15)

$$ \rho c \frac{DT}{Dt} = \rho c \left( \frac{\partial T}{\partial t} + \vec{V} \cdot \nabla T \right) = -\nabla \cdot \left( k \nabla T \right) + Q + \Phi $$  \hfill (1.16)

For the common case of constant $\mu$ it is possible to show that

$$ \nabla \cdot \left[ \mu \left( \nabla \vec{V} + \nabla \vec{V}^T \right) \right] = \mu \nabla \cdot \left( \nabla \vec{V} + \nabla \vec{V}^T \right) = \mu \Delta \vec{V} $$

where $\Delta \vec{V}$ is the Laplacian of the velocity vector, which is equal to $\nabla^2 \vec{V}$.

For constant $k$, first term on the RHS of the energy equation becomes

$$ -\nabla \cdot (k \nabla T) = -k \nabla \cdot (\nabla T) = -k \nabla^2 T $$

Using these, the final form of the governing equation set that is typically solved becomes

$$ \nabla \cdot \vec{V} = 0 $$  \hfill (1.7)

$$ \rho \left( \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} \right) = -\nabla p + \mu \nabla^2 \vec{V} + \rho \vec{f} $$  \hfill (1.17)

$$ \rho c \left( \frac{\partial T}{\partial t} + \vec{V} \cdot \nabla T \right) = -k \nabla^2 T + Q + \Phi $$  \hfill (1.18)

Eqn. (1.17) is called the **Navier-Stokes equations**. The convection term $(\vec{V} \cdot \nabla)\vec{V}$ on the left hand side of is **nonlinear**. Using the continuity equation it can also be written as $\nabla \cdot (\vec{V} \otimes \vec{V})$, where $\otimes$ is the dyadic product operator that results in a rank 2 tensor. Similarly, the convection term of the energy equation, i.e. $\vec{V} \cdot \nabla T$ term, can be written as $\nabla \cdot (\vec{V} T)$. $\mu \nabla^2 \vec{V}$ term is known as the viscous term or the diffusion term. In a sense, $\mu$ is the diffusivity of momentum. Similarly $k \nabla^2 T$ is the diffusion term of the energy equation.

For diffusion dominated flows the nonlinear convective term of the Navier-Stokes equation can be dropped and the simplified equation is called the **Stokes equations**, which is linear. Stokes equations can be used to model very low speed flows known as **creeping flows** or flows with very small length scales (micro or nano flows) where Reynolds number is small.

Heat transfer and therefore the energy equation is not always a primary concern in an incompressible flow. For isothermal (constant temperature) incompressible flows energy equation (and therefore temperature) can be dropped and only the mass and linear momentum equations are solved to obtain the velocity and pressure fields. For incompressible flows with heat effects, energy equation need to be solved to get the temperature field. But if the viscosity of the fluid can be taken as constant, energy equation decouples from the other two equations. Therefore, we can first solve the continuity and Navier-Stokes equations to find the unknown $\vec{V}$ and $p$ distribution without knowing the temperature.
After finding \( \vec{V} \), energy equation can be solved by itself to find \( T \). When the temperature dependency of viscosity is strong, all equations become coupled and need to be solved together.

### 1.3.2 Components of Governing Equations in Cartesian and Cylindrical Coordinate Systems

Eqns. (1.7), (1.17) and (1.18) are valid for any coordinate system. In order to write them for a specific coordinate system first we need to define the velocity vector components

**Cartesian**:
\[
\vec{V} = u \hat{i} + v \hat{j} + w \hat{k}
\]

**Cylindrical**:
\[
\vec{V} = V_r \hat{r} + V_\theta \hat{\theta} + V_z \hat{z}
\]

Also we need to use the following mathematical identities

**Cartesian**:
\[
\begin{align*}
\nabla A &= \frac{\partial A}{\partial x} \hat{i} + \frac{\partial A}{\partial y} \hat{j} + \frac{\partial A}{\partial z} \hat{k} \\
\nabla^2 A &= \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2} \\
\n\nabla \cdot \vec{V} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \\
\vec{V} \cdot \nabla &= u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}
\end{align*}
\]

**Cylindrical**:
\[
\begin{align*}
\nabla A &= \frac{1}{r} \frac{\partial A}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial A}{\partial \theta} \hat{\theta} + \frac{\partial A}{\partial z} \hat{z} \\
\nabla^2 A &= \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial A}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 A}{\partial \theta^2} + \frac{\partial^2 A}{\partial z^2} \\
\n\nabla \cdot \vec{V} &= \frac{1}{r} \frac{\partial}{\partial r} (r V_r) + \frac{V_\theta}{r} \frac{\partial}{\partial \theta} + \frac{V_z}{\partial z} \\
\vec{V} \cdot \nabla &= V_r \frac{\partial}{\partial r} + \frac{V_\theta}{r} \frac{\partial}{\partial \theta} + V_z \frac{\partial}{\partial z}
\end{align*}
\]

With these, governing equations for incompressible flows in Cartesian coordinate system can be obtained as follows

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (1.21)
\]

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho f_x \quad (1.22a)
\]

\[
\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \rho f_y \quad (1.22b)
\]

\[
\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \rho f_z \quad (1.22c)
\]
Viscous dissipation is given by

\[ \Phi = \mu \left\{ 2 \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right] + \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right\} \]  

(1.24)

which always has a positive value.

Incompressible flow equations in cylindrical coordinate system are

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V_r}{\partial r} \right) + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0
\]

(1.25)

\[
\rho \left( \frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_r}{\partial \theta} + V_z \frac{\partial V_r}{\partial z} - \frac{V_z^2}{r} \right) = -\frac{\partial p}{\partial r} + \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V_r}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 V_r}{\partial z^2} - \frac{2}{r} \frac{\partial V_\theta}{\partial \theta} \right] + \rho f_r
\]

(1.25a)

\[
\rho \left( \frac{\partial V_\theta}{\partial t} + V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_\theta}{\partial \theta} + V_z \frac{\partial V_\theta}{\partial z} - \frac{V_z V_\theta}{r} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V_\theta}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 V_\theta}{\partial z^2} - \frac{2}{r^2} \frac{\partial V_r}{\partial \theta} \right] + \rho f_\theta
\]

(1.25b)

\[
\rho \left( \frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_z}{\partial \theta} + V_z \frac{\partial V_z}{\partial z} - \frac{V_z^2}{r} \right) = -\frac{\partial p}{\partial z} + \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V_z}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 V_z}{\partial z^2} \right] + \rho f_z
\]

(1.25c)

\[
\rho c_p \left( \frac{\partial T}{\partial t} + V_r \frac{\partial T}{\partial r} + \frac{V_\theta}{r} \frac{\partial T}{\partial \theta} + V_z \frac{\partial T}{\partial z} \right) = k \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 T}{\partial z^2} \right] + \Phi
\]

(1.26)

### 1.3.3 Nondimensionalization of Governing Equations

It is possible, and sometimes preferable, to write governing equations in nondimensional form. To do this we need to select a characteristic quantities that describe the flow problem, such as a characteristic length \( L \), characteristic velocity \( U_\infty \), characteristic pressure \( p_\infty \) and characteristic temperature \( T_\infty \). Using these characteristic quantities, the following nondimensional parameters can be defined

\[
t^* = \frac{t}{L/U_\infty}, \quad x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad z^* = \frac{z}{L}, \quad u^* = \frac{u}{U_\infty}, \quad v^* = \frac{v}{U_\infty}, \quad w^* = \frac{w}{U_\infty}, \quad \ldots
\]

\[
\ldots, \quad p^* = \frac{p - p_\infty}{\rho U_\infty^2}, \quad T^* = \frac{T - T_\infty}{\Delta T}
\]

(1.29)

where \( \Delta T \) is a known reference temperature difference in the flow field such as the one between a constant wall temperature (if it exists) and a surrounding temperature \( T_\infty \) (if it exists). Note that these nondimensionalizations are not unique and can be done in other ways. Using these definitions in the governing equations, their nondimensional forms can be obtained. For example for an incompressible flow without body forces, equations (1.22)-(1.24) can be converted into the following ones

\[
\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} + w^* \frac{\partial u^*}{\partial z^*} = \frac{\partial p^*}{\partial x^*} + \frac{1}{Re} \left( \frac{\partial^2 u^*}{\partial x^*^2} + \frac{\partial^2 u^*}{\partial y^*^2} + \frac{\partial^2 u^*}{\partial z^*^2} \right)
\]

(1.31a)

\[
\frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} + w^* \frac{\partial v^*}{\partial z^*} = \frac{\partial p^*}{\partial y^*} + \frac{1}{Re} \left( \frac{\partial^2 v^*}{\partial x^*^2} + \frac{\partial^2 v^*}{\partial y^*^2} + \frac{\partial^2 v^*}{\partial z^*^2} \right)
\]

(1.31b)
\[
\frac{\partial w^*}{\partial t^*} + u^* \frac{\partial w^*}{\partial x^*} + v^* \frac{\partial w^*}{\partial y^*} + w^* \frac{\partial w^*}{\partial z^*} = \frac{\partial p^*}{\partial z^*} + \frac{1}{Re} \left( \frac{\partial^2 w^*}{\partial x^*^2} + \frac{\partial^2 w^*}{\partial y^*^2} + \frac{\partial^2 w^*}{\partial z^*^2} \right)
\]  
(1.31c)

\[
\frac{\partial T^*}{\partial t^*} + u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} + w^* \frac{\partial T^*}{\partial z^*} = \frac{1}{RePr} \left( \frac{\partial^2 T^*}{\partial x^*^2} + \frac{\partial^2 T^*}{\partial y^*^2} + \frac{\partial^2 T^*}{\partial z^*^2} \right) + \frac{Ec}{Re} \phi^*
\]  
(1.32)

As seen, the equations are very similar to their dimensional counterparts with additional nondimensional numbers. Reynolds number \((Re = U_\infty L/\nu)\) is seen in the momentum and energy equations. In the energy equation we also have Prandtl and Eckert numbers.

Reynolds number is a measure of the balance between convective and viscous (diffusive) terms of the Navier-Stokes equation. High and low \(Re\) flows are said to be convection and diffusion dominated, respectively. The nondimensional time given in equation (1.29) is suitable for convection dominated (high \(Re\)) flows. For diffusion dominated problems using a diffusion time scale as \(t^* = \frac{t}{L^2/\nu}\) is more suitable. In this case, the form of the nondimensional momentum equations will be different.

Eckert number \((Ec = U_\infty L/(c_p \Delta T))\) is the ratio of flow's kinetic energy to a representative enthalpy difference. As it gets larger the importance of viscous dissipation is amplified. Prandtl number \((Pr = \nu/\alpha = \nu/(k/c_p))\) is the ratio of momentum and thermal diffusivities. As it gets larger the importance of diffusion term on the right hand side of equation (1.32) diminishes and the convective heat transfer modeled by the terms on the left hand side becomes more dominant. Multiplication of Reynolds and Prandtl numbers is called the Peclet number \((Pe)\).

There are many other important nondimensional numbers in fluid mechanics and heat transfer. Some of them appear in the equations similar to the ones mentioned above (such as the Grashof number seen in natural convection flows), and some other nondimensional numbers appear inside the boundary conditions (such as the Nusselt number).

### 1.3.4 Turbulence Modeling

One of today’s most important challenges for the numerical solution of fluid flow problems is the modeling and simulation of turbulence. Although Navier-Stokes equations are believed to be capable of describing turbulent flows in full detail, with today’s computational resources it is simply impossible to have simulations that will yield all the details of a turbulent flow in a 3D problem domain for a realistically high Reynolds number. The difficulty is due to the fact that turbulent flows have a wide range of length and time scales. We need to resolve both large and very small flow features that behave quite differently in time. Therefore, a computational study is restricted to use extremely fine meshes and extremely small time steps, which exceeds today’s computational power. As the Reynolds number increases, the range covered by the minimum and maximum length and time scales increase, making the numerical solution more challenging.

Solving the governing flow equations with very fine computational grids and very small time steps to get the whole detail of a turbulent flow is called Direct Numerical Simulation (DNS). DNS involves no turbulence modeling. Today it is a very active research area. However, for practical real world problems it is too restrictive computationally, even on the most powerful supercomputers. Large Eddy Simulation (LES) is the next best one after DNS in terms of physical correctness and numerical accuracy. LES is based on spatial filtering of turbulent flow features, where the large ones are solved numerically and small ones are modeled. Since there is no need to resolve (capture) the small flow features, the computational mesh does need to be as fine as required by DNS. This makes LES much cheaper than DNS, but still too costly in many cases for a practicing engineer who needs to perform flow simulations as an everyday practice.
The third and the most commonly used option in the industry is the Reynolds Averaged Navier-Stokes (RANS) type modeling, which is based on a filtering in time. Flow variables such as velocity and pressure are decomposed into time averaged (mean) and fluctuating parts and mean flow equations are solved and the effects of turbulent fluctuations on the averaged quantities are modeled by the solution of extra transport equations. Computational demand of RANS is much less compared to LES, but this comes with the price of reduced accuracy and increased modeling uncertainty. Although certain RANS models are clearly superior to others in terms of the correct physics they involve, unfortunately none of them can predict all types of turbulent flows more accurately and efficiently than the others. They all contain empirical constants, which are tuned numbers such that simulation results fit to a set of selected experimental results.

1.3.5 Model Differential Equations and the Advection-Diffusion Equation (ADE)

Governing equations of fluid mechanics and heat transfer problem are usually PDEs of second in space and first order in time. There are more than one unknown and coupled PDEs need to be solved simultaneously. Also Navier-Stokes equations are nonlinear. In short, these equations are not the most appropriate ones for learning the basics of a numerical technique, such as the FEM. Instead we prefer to start with simplified model ODEs and PDEs. It is possible to construct problems governed by them with known analytical solutions so that numerical codes can be validated easily. Although these model equations are much simpler to solve compared to the actual governing equations, they still give us a chance to get an experience about the numerical difficulties that we’ll face with when we start working with more realistic ones.

Among the many available model equations, the following Advection-Diffusion Equation (ADE), which is also called the Generic Transport Equation (GTE), will be used extensively in this course

\[
\frac{\partial \phi}{\partial t} + \vec{V} \cdot \nabla \phi = \alpha \nabla^2 \phi + S
\]  

(1.33)

In Cartesian coordinate system it becomes

\[
\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} + w \frac{\partial \phi}{\partial z} = \alpha \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right) + S
\]  

(1.34)

ADE is the simplest model equation that can be used to test the performance of different numerical schemes for problems involving advection (convection) and diffusion phenomena. It can be seen as a linearized and simplified scalar form of the Navier-Stokes equation with a single unknown \( \phi \). It is also very similar to the energy conservation equation. The scalar unknown \( \phi \) is advected (convected) with a known velocity field \( \vec{V} \), which can be taken to be divergence free to satisfy the continuity equation of incompressible flows. At the same time \( \phi \) is diffused with a known constant and isotropic diffusivity \( \alpha \). \( S \) represents the known source term, which can be zero.

In this course we’ll use the ADE extensively to study the difficulties faced with convection dominated problems. Note that for a special case of no velocity (\( \vec{V} = 0 \), pure diffusion), we obtain the following unsteady heat equation, which is parabolic in nature.

\[
\frac{\partial \phi}{\partial t} = \alpha \nabla^2 \phi + S
\]  

(1.35)

For the following pure advection case (\( \alpha = 0 \)) the equation becomes hyperbolic in nature.

\[
\frac{\partial \phi}{\partial t} + \vec{V} \cdot \nabla \phi = S
\]  

(1.36)
For $\vec{V} = 0$ and $\frac{\partial \phi}{\partial t} = 0$, we get the following steady Poisson equation (steady heat equation) which is elliptic in nature.

$$-\alpha \nabla^2 \phi = S$$  \hspace{1cm} (1.37)

To discuss the dimensionless form of the ADE, let’s use the following 1D form and neglect the source term for simplicity

$$\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} = \alpha \frac{\partial^2 \phi}{\partial x^2}$$  \hspace{1cm} (1.38)

Using a characteristic length $L$, a characteristic velocity $U$ and a characteristic time $L/U$, following dimensionless variables can be defined

$$\phi^* = \frac{\phi}{\Delta \phi}, \quad x^* = \frac{x}{L}, \quad u^* = \frac{u}{U}, \quad t^* = \frac{t}{L/U}$$  \hspace{1cm} (1.39)

where $\Delta \phi$ is a characteristic $\phi$ difference used in defining the problem. Using these dimensionless variables, the following nondimensional form of the ADE can be derived

$$\frac{\partial \phi^*}{\partial t^*} + u^* \frac{\partial \phi^*}{\partial x^*} = \frac{1}{Pe} \frac{\partial^2 \phi^*}{\partial x^{*2}}$$  \hspace{1cm} (1.40)

where the nondimensional Peclet number (Pe) is defined as

$$Pe = \frac{UL}{\alpha}$$  \hspace{1cm} (1.41)

It represents a ratio between the “strength” of advection and diffusion processes. Advection dominated flows are characterized by high $Pe$ values. Note that for diffusion dominated problems using a characteristic time of $L^2/\alpha$ is more appropriate.

### 1.3.6 Mathematical and Physical Classification of PDEs

It is possible to classify PDEs in three categories

- Elliptic
- Parabolic
- Hyperbolic

This classification is related to the "characteristics" of PDEs. Characteristics are paths (curved surfaces in $xyzt$ hyperspace in general) in the solution domain along which information propagates. If a PDE possesses real characteristics, then information propagates along these characteristics. If no real characteristics exist, then there are no preferred paths of information propagation. The presence or absence of real characteristics has a significant impact on the solution of a PDE, both analytically and numerically.

Mathematical procedure of identifying the type of a PDE depends on the mathematical details such as the order of the PDE and it can be studied from Hoffmann [2] and Hoffmann and Chiang [3] (related chapters of these two references are available as PDF files at the Files tab of the course web site). Here let’s concentrate on the difference of the physics of the problems that are governed by different types of PDEs.
Parabolic and hyperbolic PDEs have real characteristics. Problems governed by these two types of PDEs are called propagation problems, which are actually initial value problems in which the solution starts from a known initial condition and propagates in time. In the meantime, the solution is guided by the boundary conditions. Solution domains of parabolic and hyperbolic PDEs are said to be open in the sense that in theory the solution may continue infinitely long in the time domain.

Unsteady heat equation given in Eqn. (1.35) is a model parabolic PDE. For simplicity consider its 1D version, which can be used to study the temperature distribution of a bar which is left to cool down from a known initial temperature distribution. We are interested in the temperature distribution at various stages of this cooling. To solve this problem, boundary conditions need to be specified at both ends of the bar, e.g. two ends are kept at fixed temperatures. Starting from the known initial temperature distribution, new temperature values at different time levels can be calculated numerically. In practice the solution does not continue infinitely long in time, but it ends at a proper final time, e.g. when the process reaches steady-state, if such a state exists, where the temperature of the bar no longer changes.

As said before, parabolic PDEs have real characteristics. From a physical standpoint this divides the solution domain into a zone of dependence and a zone of influence as seen in the left plot of the following figure. In the "cooling of a bar" problem mentioned above consider the mid-point P of the bar. Initially this point has a certain temperature and this temperature value affects the temperature of all points on the bar at all future times. That is, if we start the solution with a different temperature at point P, temperature all along the bar at all time levels will change. Therefore, at the beginning of a solution the whole problem domain in $xt$ plane is the zone of influence for point P. Now consider the same point P at $t = 5$ s after cooling starts. Point P has a certain temperature value at $t = 5$ s, but this value can affect only the solution ahead in time, i.e. it can only affect the solution between $5 < t < \infty$. Now the part of $xt$ plane corresponding to $0 < t < 5$ is the zone of dependence and the part $5 < t < \infty$ is the zone of influence. The solution at point P at $t = 5$ depends on the solution in the zone of dependence and will affect the solution in the zone of influence.

Hyperbolic PDEs also have real characteristics and they also govern propagation problems. Consider the following 1D wave equation

Figure. Domain of dependence and domain and influence for a parabolic PDE (left) and a hyperbolic PDE (right) [2]
\[ \frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2} \]

It is used to study the travel of a pressure disturbance (acoustic pressure) with a known initial shape in a 1D domain with a constant speed of sound \( c \). The value of \( c \) determines how fast the information can propagate in the solution domain. Since \( c \) has a finite value, the solution at a certain point \( P \) of the \( xt \) plane can only affect certain parts of the future solutions. Similarly, it can be affected by only certain parts of the previous solutions. The lines drawn in the right plot of the above figure are called characteristic lines. These are the lines along which information, i.e. the pressure disturbance, travels. The slopes of these lines are determined by the constant speed \( c \) of the pressure wave. As the wave speed increases the slopes of these lines change and in the limiting case of infinite wave speed the characteristic lines will be parallel to the \( x \) axis, which is the case for the previously discussed parabolic PDEs. So for a parabolic PDE information travels with infinite speed and can reach and influence all spatial points of the problem domain immediately.

Elliptic PDEs govern equilibrium (not propagation) problems, which are used to obtain steady state solutions in closed spatial domains. Poisson equation given in the previous section that for example can be used to calculate the temperature distribution over a square plate heated at the center with a known heat source. Here the problem domain is in the \( xy \) plane and after maybe a small transition period the problem can reach a steady state equilibrium and time is no longer a variable. Depending on the amount of heat source and the boundary conditions specified at the four edges of the plate a certain steady-state temperature distribution can be calculated. Elliptic PDEs have no real characteristics and both the domain of dependence and the domain of influence is the whole problem domain for all points. Solution at every point of the problem domain is influenced by the solution at all other points.