Coordinate Systems, Transformations and their Line-Surface-Volume Elements

Coordinate Systems:
Three most common coordinate systems used in 3-dimensional representations are:

a) Cartesian coordinates
b) Cylindrical (polar) coordinates
c) Spherical (polar) coordinates

Their basic coordinates and associated unit vectors are shown in Figures 1, 2 and 3.

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**Figure 1  Cartesian Coordinate System**

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**Figure 2  Cylindrical Coordinate System**

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\( \vec{e}_r \): radially outward from the center of the cylinder (Oz), on \( z = \) constant plane

\( \vec{e}_\theta \): tangent to the cylinder lateral surface in counterclockwise direction, on \( z = \) constant plane

\( \vec{e}_z \): in +z direction
Spherical coordinates

\( \mathbf{\hat{e}}_\rho \): radially outward from the center of the sphere, i.e. the origin
\( \mathbf{\hat{e}}_\theta \): tangent to the sphere in counterclockwise direction, on \( z = \text{constant} \) plane
\( \mathbf{\hat{e}}_\phi \): tangent to the sphere in clockwise direction on \( \theta = \text{constant} \) plane

**Figure 3  Spherical Coordinate System**

In Cartesian coordinates,
\( x = \text{constant} \in [\infty, +\infty] \) represents a plane parallel to Oyz plane,
\( y = \text{constant} \in [\infty, +\infty] \) represents a plane parallel to Oxz plane,
\( z = \text{constant} \in [\infty, +\infty] \) represents a plane parallel to Oxy plane.

In cylindrical coordinates,
\( r = \text{constant} \in [0, +\infty] \) represents a cylinder whose axis is Oz and radius is \( r \),
\( \theta = \text{constant} \in [0, +2\pi] \) represents a vertical half plane containing Oz,
\( z = \text{constant} \in [\infty, +\infty] \) represents a plane parallel to Oxy plane.

In spherical coordinates,
\( \rho = \text{constant} \in [0, +\infty] \) represents a sphere whose center is O and radius is \( \rho \),
\( \phi = \text{constant} \in [0, +\pi] \) represents a cone whose axis is Oz and tip is located at O,
\( \theta = \text{constant} \in [0, +2\pi] \) represents a vertical half plane containing Oz.

**Transformations between Coordinate Systems:**

<table>
<thead>
<tr>
<th>Cylindrical to Cartesian coordinates</th>
<th>Cartesian to cylindrical coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = r \cos \theta )</td>
<td>( r = \sqrt{x^2 + y^2} )</td>
</tr>
<tr>
<td>( y = r \sin \theta )</td>
<td>( \theta = \tan^{-1}(y / x) )</td>
</tr>
<tr>
<td>( z = z )</td>
<td>( z = z )</td>
</tr>
</tbody>
</table>
### Spherical to Cartesian coordinates

\[
\begin{align*}
\rho &= \sqrt{x^2 + y^2 + z^2} \\
\theta &= \tan^{-1}(y/x) \\
\phi &= \tan^{-1}\left(\sqrt{\frac{x^2 + y^2}{z}}\right)
\end{align*}
\]

### Cartesian to spherical coordinates

\[
\begin{align*}
x &= \rho \sin \phi \cos \theta \\
y &= \rho \sin \phi \sin \theta \\
z &= \rho \cos \phi
\end{align*}
\]

### Spherical to cylindrical coordinates

\[
\begin{align*}
r &= \rho \sin \phi \\
z &= \rho \cos \phi \\
\theta &= \theta
\end{align*}
\]

### Cylindrical to spherical coordinates

\[
\begin{align*}
\rho &= \sqrt{r^2 + z^2} \\
\phi &= \tan^{-1}(r/z) \\
\theta &= \theta
\end{align*}
\]

### Line elements:

In Cartesian coordinates:

\[
ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}
\]

In cylindrical coordinates:

\[
ds = \sqrt{(dr)^2 + r^2 (d\theta)^2 + (dz)^2}
\]

In spherical coordinates:

\[
ds = \sqrt{(dp)^2 + \rho^2 (d\phi)^2 + \rho^2 (\sin \phi)^2 (d\theta)^2}
\]

### Surface and volume elements:

In Cartesian coordinates:

\[
\begin{align*}
dS_1 &= dy \, dz \\
dS_2 &= dx \, dz \\
dS_3 &= dx \, dy
\end{align*}
\]

**Volume element:**

\[dV = dx \, dy \, dz\]
In cylindrical coordinates:

Surface elements:
\[ dS_1 = dr \, dz \quad ; \quad dS_2 = r \, d\theta \, dz \quad ; \quad dS_3 = r \, dr \, d\theta \]

Volume element:
\[ dV = r \, dr \, d\theta \, dz \]

In spherical coordinates:

Surface elements:
\[ dS_1 = \rho \, d\rho \, d\phi \quad ; \quad dS_2 = \rho^2 \, \sin \phi \, d\theta \, d\phi \quad ; \quad dS_3 = \rho \, \sin \phi \, d\rho \, d\theta \]

Volume element:
\[ dV = \rho^2 \, \sin \phi \, d\rho \, d\theta \, d\phi \]